ST109 Class 10 of Week 10

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Dec 5, 2024

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1 Discrete Multivariate Random Variable

Joint Probability Function

Suppose that X and Y are discrete random variables.

Joint Probability Function

$$f_{X,Y}(x, y) = \mathbb{P}(X = x, Y = y)$$

Properties of Joint PF

1. $f(x, y) \geq 0$ for all (x, y)
2. $\sum_x \sum_y p(x, y) = 1$

A Univariate View

Marginal Distribution

$$f_X(x) = \sum_y f(x, y) \qquad f_Y(y) = \sum_x f(x, y)$$

Expectation and Variance

$$\mathbb{E}(X) = \sum_x x f_X(x) \qquad \mathbb{E}(Y) = \sum_y y f_Y(y)$$

$$\text{var}(X) = \sum_x x^2 f_X(x) - [\mathbb{E}(X)]^2 \qquad \text{var}(Y) = \sum_y y^2 f_Y(y) - [\mathbb{E}(Y)]^2$$

A Multivariate View

Conditional Probability Function

The conditional distribution of Y given $X = x$ is the discrete probability distribution with the pf:

$$f_{Y|X}(y \mid x) = \mathbb{P}(Y = y \mid X = x) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(X = x)} = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

for any value y. Similar for the conditional pf of X given $Y = y$:

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

Conditional Expectation and Conditional Variance

$$\mathbb{E}_{Y|X}(Y \mid x) = \sum_y y f_{Y|X}(y \mid x) \qquad \mathbb{E}_{X|Y}(X \mid y) = \sum_x x f_{X|Y}(x \mid y)$$

A Multivariate View (Cont.)

Covariance

Measures the strength of a linear association between X and Y:

$$\operatorname{cov}(X, Y) = \mathbb{E}[X - \mathbb{E}(X)][Y - \mathbb{E}(Y)] = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

Properties of Covariance

1. $\operatorname{cov}(c, X) = 0$, where c is a constant
2. $\operatorname{cov}(aX + b, cY + d) = ac \operatorname{cov}(X, Y)$
3. $\operatorname{cov}(X + Y, Z) = \operatorname{cov}(X, Z) + \operatorname{cov}(Y, Z)$, where Z is a random variable

A Multivariate View (Cont.)

Correlation

Also measures the strength of a linear association between X and Y, but standardized:

$$\operatorname{corr}(X, Y) = \frac{\operatorname{cov}(X, Y)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}}$$

Properties of Correlation

1. $\operatorname{corr}(X, Y) \in [-1, 1]$
2. $\operatorname{corr}(c, X) = 0$, where c is a constant
3. $\operatorname{corr}(aX + b, cY + d) = \operatorname{sign}(ac)\operatorname{corr}(X, Y)$, where $\operatorname{sign}(\cdot)$ takes the sign of $(\cdot)$
4. $\operatorname{corr}(X + Y, Z) \neq \operatorname{corr}(X, Z) + \operatorname{corr}(Y, Z)$ in most case

A Multivariate View (Cont.)

Interpretation of Correlation

- If $\text{corr}(X, Y) = 1$ (or -1), then X is a linear function of Y, i.e. $X = aY + b$, with $a > 0$ (or $a < 0$).
- If $\text{corr}(X, Y) > 0$ (or < 0), then X and Y are positively (or negatively) correlated.
- If $\text{corr}(X, Y) = 0$, then X and Y are uncorrelated.

Remark!!

$$\text{Independent} \implies \text{Uncorrelated}$$
$$\textbf{Uncorrelated} \not\Longrightarrow \textbf{Independent}$$

that is, independence is a **stronger** argument than correlation.