ST304 Week 3

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Autocovariance and autocorrelation (weak stationarity)

Weak stationarity $\Rightarrow$ dependence only on lag

If (X_t) is weakly stationary, then $E(X_t) = \mu$ is constant and

$$\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(X_{s+h}, X_{t+h}) \quad \Rightarrow \quad \mathrm{Cov}(X_s, X_t) = \gamma(t-s).$$

Thus for lag $h = t - s \in \mathbb{Z}$,

$$\gamma(h) := \mathrm{Cov}(X_s, X_t).$$

Autocorrelation function (scale-free)

$$\rho(h) := \mathrm{Corr}(X_s, X_t) = \frac{\gamma(h)}{\gamma(0)}, \qquad \rho(0) = 1.$$

$\rho(h)$ measures linear dependence at lag h on a common scale.

Properties of ACVF and ACF

Let (X_t) be weakly stationary with ACVF $\gamma(h)$ and ACF $\rho(h)$.

Basic properties

- $\gamma(0) = \mathrm{Var}(X_t) \in [0, \infty)$ and $\rho(0) = 1$.
- Symmetry: $\gamma(h) = \gamma(-h)$ and $\rho(h) = \rho(-h)$.
- Cauchy–Schwarz bound: $|\gamma(h)| \leq \gamma(0)$, so $|\rho(h)| \leq 1$.

Non-negative definiteness

For any $m \in \mathbb{N}$, times $t_1, \dots, t_m$ and coefficients $a_1, \dots, a_m$,

$$\sum_{j=1}^{m} \sum_{k=1}^{m} a_j \, \gamma(t_j - t_k) \, a_k \; = \; \mathrm{Var}\Big(\sum_{j=1}^{m} a_j X_{t_j} \Big) \; \geq \; 0.$$

Sample autocovariance and autocorrelation

Suppose we observe $X_1, \dots, X_n$ from a weakly stationary process. Let $\bar{X}_n := \frac{1}{n}\sum_{t=1}^{n} X_t$.

Sample autocovariance function (ACVF)

$$\hat{\gamma}_n(h) := \frac{1}{n}\sum_{t=1}^{n-|h|}(X_t - \bar{X}_n)(X_{t+|h|} - \bar{X}_n).$$

Sample autocorrelation function (ACF)

$$\hat{\rho}_n(h) := \frac{\hat{\gamma}_n(h)}{\hat{\gamma}_n(0)}.$$

Note $\hat{\gamma}_n(0) = \frac{1}{n}\sum_{t=1}^{n}(X_t - \bar{X}_n)^2$ (sample variance).

Estimators: unbiased vs practical choice

- If the true mean μ is known, then for $|h| \le n-1$,

$$\tilde{\gamma}_n^*(h) := \frac{1}{n-|h|} \sum_{t=1}^{n-|h|} (X_t - \mu)(X_{t+|h|} - \mu)$$

  is an **unbiased** estimator of $\gamma(h)$.
- In practice μ is unknown, so we replace it by $\bar{X}_n$:

$$\tilde{\gamma}_n(h) := \frac{1}{n-|h|} \sum_{t=1}^{n-|h|} (X_t - \bar{X}_n)(X_{t+|h|} - \bar{X}_n),$$

  which is generally **biased**.
- The notes often prefer using $\hat{\gamma}_n(h)$ with denominator n: despite bias, it often has **smaller MSE** and yields a **non-negative definite** ACVF.

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Moving average (MA) processes and ACF

Definition (MA(q))

A MA process of order $q \in \mathbb{N}$ is

$$X_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}, \quad \varepsilon_t \sim WN(0, \sigma^2).$$

ACVF/ACF behaviour

MA(q) is weakly stationary and its ACVF satisfies

$$\gamma(h) = \begin{cases} \sigma^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|}, & |h| \le q, \\ 0, & |h| > q. \end{cases}$$

Hence the ACF **cuts off**: $\rho(h) = 0$ for all $|h| > q$.

Autoregressive (AR) processes and ACF

Definition (AR(p))

An AR process of order $p \in \mathbb{N}$ satisfies

$$X_t - \mu = \phi_1(X_{t-1} - \mu) + \cdots + \phi_p(X_{t-p} - \mu) + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma^2).$$

ACF behaviour: AR tails

Unlike MA(q), an AR(p) generally has an ACF that **does not cut off**.

- The ACF typically **decays** with lag (often exponentially / as damped oscillations).
- For AR(1) with $|\phi_1| < 1$, the stationary solution has

$$\rho(h) = \phi_1^{|h|},$$

so positive ϕ_1 gives monotone decay, while negative ϕ_1 alternates sign.