# ST304 Week 4

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# Causality

## Definition (Causality)

A weakly stationary process $(X_t)$ is **causal** (with respect to a white noise $\varepsilon_t \sim WN(0, \sigma^2)$) if it admits a representation

$$X_t = \sum_{j=0}^{\infty} \psi_j \, \varepsilon_{t-j}, \qquad \sum_{j=0}^{\infty} |\psi_j| < \infty.$$

## Interpretation

- **One-sided:** $X_t$ depends only on **current and past** shocks.
- **Forecasting:** $X_t$ is driven by information available up to time $t$.
- **Avoid AR redundancy:** the same AR equation can admit a **stationary but non-causal** solution (depends on future shocks), so we restrict to the **causal** solution to get a canonical, interpretable model.

# Stationarity vs Causality (Example: AR(1))

## AR(1) and its polynomial

$$X_t = \phi_1 X_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim WN(0, \sigma^2),$$

and the AR polynomial is $\Phi(z) = 1 - \phi_1 z$. Its (only) root is $z^\star = 1/\phi_1$.

## Key conditions

- **Stationary** $\iff$ $|z^\star| \neq 1$ (no unit root) $\iff$ $|\phi_1| \neq 1$.
- **Causal** $\iff$ $|z^\star| > 1$ $\iff$ $|\phi_1| < 1$.
- If $|z^\star| < 1$ (i.e. $|\phi_1| > 1$): a stationary solution exists but is **non-causal**. Start from $X_t = \phi_1^{-1} X_{t+1} - \phi_1^{-1} \varepsilon_{t+1}$, we have

$$X_t = -\sum_{j=1}^{\infty} \phi_1^{-j} \varepsilon_{t+j}.$$

# Invertibility

## Definition (Invertibility)

A weakly stationary process $(X_t)$ is **invertible** w.r.t. $\varepsilon_t \sim WN(0, \sigma^2)$ if

$$\varepsilon_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}, \qquad \sum_{j=0}^{\infty} |\pi_j| < \infty.$$

## Interpretation

- **One-sided:** using only current/past $X_t, X_{t-1}, \ldots$.
- **Avoid MA redundancy:** different MA coefficients can produce the **same** ACF/ACVF, so we choose the **invertible** representation to make parameters identifiable.

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# ARMA$(p, q)$

**Definition (ARMA$(p, q)$)**

For $p, q \in \mathbb{N}_0$ (assume $\mu = 0$),

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}, \quad \varepsilon_t \sim WN(0, \sigma^2).$$

**Definition (AR and MA polynomials; backshift form)**

Define the AR and MA polynomials

$$\Phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p, \qquad \Theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q,$$

and let $B$ be the backshift operator, $BX_t = X_{t-1}$. Then the ARMA$(p, q)$ model can be written as

$$\Phi(B) X_t = \Theta(B) \varepsilon_t.$$

# Causality & invertibility for ARMA

**Avoid over-parametrisation**

We typically require $\Phi$ and $\Theta$ have **no common complex roots** (otherwise factors can "cancel" and the model is not identifiable).

**Theorem (root conditions)**

Consider an ARMA$(p, q)$ with $\Phi$ and $\Theta$ sharing no roots.

- **Stationary solution exists** $\iff$ $\Phi$ has **no unit roots**: all roots satisfy $|w| \neq 1$.
- **Causal stationary solution** $\iff$ all roots of $\Phi$ satisfy $|w| > 1$ (outside the unit disc).
- **Invertible** $\iff$ all roots of $\Theta$ satisfy $|w| > 1$.

# ARMA($p, q$): stationarity, causality, invertibility

**Assume $\Phi$ and $\Theta$ share no common complex roots**

| **Property** | $\Phi(z)$ **(AR part)** | $\Theta(z)$ **(MA part)** |
|---|---|---|
| Stationary | **No unit roots**: $\lvert w \rvert \neq 1$ for all roots. | – |
| Causal | **All roots outside unit disc**: $\lvert w \rvert > 1$ for all roots. | – |
| Invertible | – | **All roots outside unit disc**: $\lvert w \rvert > 1$ for all roots. |