ST326 Week 10

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Dec 5, 2025

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CAPM (One factor model)

Denote r_i to be the return r.v. of the i-th risky asset, r_{mkt} to be the market return r.v., and r_f to be the rist-free return. In addition, $\mu_i = \mathbb{E}(r_i)$ and $\mu_{mkt} = \mathbb{E}(r_{mkt})$.

Capital Asset Pricing Model (CAPM)

One consequence of the one fund theorem is the Capital Asset Pricing Model:

$$\mu_i - r_f = \beta_i(\mu_{mkt} - r_f),$$

and that

$$\beta_i = \frac{\text{Cov}(r_i, r_{mkt})}{\text{Var}(r_{mkt})}$$

CAPM (Cont.)

The value of β_i means that we can rewrite CAPM into

$$r_i - r_f = \beta_i(r_{mkt} - r_f) + \varepsilon_i,$$

where $\mathbb{E}(\varepsilon_i) = 0$ and $\mathrm{Cov}(r_{mkt}, \varepsilon_i) = 0$. By taking Var on both side, we have

$$\mathrm{Var}(r_i) = \underbrace{\beta_i^2\,\mathrm{Var}(r_{mkt})}_{\text{systematic risk}} + \underbrace{\mathrm{Var}(\varepsilon_i)}_{idiosyncratic risk}.$$

We say the i-th asset is

- **aggressive** if $\beta_i > 1$;
- **neutral** if $\beta_i = 1$;
- **defensive** if $\beta_i < 1$.

Multifactor model (Non-examinable)

The CAPM is an example of a one-factor pricing model for the return. The multifactor pricing model which is assumed in arbitrage pricing theory is of the form

$$r_{it} = \alpha_i + \boldsymbol{\beta}_i^\top \mathbf{f}_t + \varepsilon_{it}, \qquad i = 1, \ldots, p$$

where $\boldsymbol{\beta}_i, \mathbf{f}_t \in \mathbb{R}^K$, that $\mathbf{f}_t$ denotes a vector of K factors, and $\boldsymbol{\beta}_i$ denotes the factor loadings of the i-th asset. In matrix form, the multifactor pricing model can be expressed as

$$\mathbf{r}_t = \boldsymbol{\alpha} + \mathbf{B}\mathbf{f}_t + \boldsymbol{\varepsilon}_t,$$

where $\mathbf{B} = (\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_p)^\top$ denotes the factor loading matrix.
Remarks:

- $\mathbf{f}_t$ does not depend on i, and hence they are the same for each r_{it}, which is why the term factor, since they exists in each r_{it} through the factor loading matrix $\mathbf{B}$.
- When $K = 1$, it is the CAPM, the one-factor model.

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Latent factor model

Consider the multifactor model

$$\mathbf{r}_t = \boldsymbol{\alpha} + \mathbf{B}\mathbf{f}_t + \boldsymbol{\varepsilon}_t.$$

If $\mathbf{f}_t$ is unknown, then it is called a latent factor model. One of the purpose of this model is **dimension reduction**, which simplifies the calculation of $\boldsymbol{\Sigma}^{-1}$, which is being exhaustively calculated in portfolio allocation when p is large. Assume the number of factors K is fixed, and $K << p$, so that the original dimension p is being reduced to a much smaller K.

Latent factor model (Cont.)

Consider we have observations $\{\mathbf{r}_t\}_{t=1}^T$. Assume K is fixed. Under some identification condition $\mathbf{B}^\top\mathbf{B} = I_K$ and $\bar{\mathbf{f}}_t = \mathbf{0}$, the model

$$\mathbf{r}_t = \boldsymbol{\alpha} + \mathbf{B}\mathbf{f}_t + \boldsymbol{\varepsilon}_t$$

has the least square estimation $(\min\{T^{-1}\sum_{t=1}^T \|\mathbf{r}_t - \boldsymbol{\alpha} - \mathbf{B}\mathbf{f}_t\|^2\})$

$$\begin{aligned}
\widehat{\boldsymbol{\alpha}} &= \bar{\mathbf{r}} = T^{-1}\sum_{t=1}^{T}\mathbf{r}_t, \\
\widehat{\mathbf{f}}_t &= \mathbf{B}^\top(\mathbf{r}_t - \bar{\mathbf{r}}), \\
\widehat{\boldsymbol{\varepsilon}}_t &= \mathbf{r}_t - \widehat{\boldsymbol{\alpha}} - \widehat{\mathbf{B}}\widehat{\mathbf{f}}_t
\end{aligned}$$

and $\widehat{\mathbf{B}}$ is the K eigenvectors corresponding to the K largest eigenvalues of the sample covariance matrix $\widetilde{\boldsymbol{\Sigma}}_r$, where

$$\widetilde{\boldsymbol{\Sigma}}_r = T^{-1}\sum_{t=1}^{T}(\mathbf{r}_t - \bar{\mathbf{r}})(\mathbf{r}_t - \bar{\mathbf{r}})^\top.$$

Latent factor model (Cont.)

Assume $\text{Cov}(\mathbf{f}_t, \varepsilon_t) = 0$. Since one of our goal is to calculate $\mathbf{\Sigma}_r = \text{Var}(\mathbf{r}_t)$, consider the variance decomposition again:

$$\text{Var}(\mathbf{r}_t) = \mathbf{B}\mathbf{\Sigma}_f\mathbf{B}^\top + \mathbf{\Sigma}_\varepsilon.$$

We therefore have an estimator for $\mathbf{\Sigma}_r$:

$$\widehat{\mathbf{\Sigma}}_r = \widehat{\mathbf{B}}\widehat{\mathbf{\Sigma}}_f\widehat{\mathbf{B}}^\top + \widehat{\mathbf{\Sigma}}_\varepsilon.$$

If we directly calculate the sample covariance of $\widehat{\varepsilon}_t$, there are $p(p+1)/2$ parameters need to be calculated, which did not simplify our problem.

- **Strict factor model**: assume $\mathbf{\Sigma}_\varepsilon$ is diagonal. There are p parameters need to be calculated.

$$\widehat{\mathbf{\Sigma}}_\varepsilon^S = \text{diag}\Big(T^{-1}\sum_{t=1}^{T}(\mathbf{r}_t - \boldsymbol{\alpha} - \mathbf{B}\mathbf{f}_t)(\mathbf{r}_t - \boldsymbol{\alpha} - \mathbf{B}\mathbf{f}_t)^\top\Big).$$

- **Approximate factor model**: assume $\mathbf{\Sigma}_\varepsilon$ is sparse, i.e., contains lots of zeros. See estimation procedure by Fan et al., 2013 [1] as a start.

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References

Fan, J., Liao, Y., and Mincheva, M.
Large covariance estimation by thresholding principal orthogonal complements.
Journal of the Royal Statistical Society: Series B (Statistical Methodology) 75, 4 (2013), 603–680.