ST326 Week 11

Kaixin Liu[1]

[1]PhD Student in Statistics, LSE

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Value-at-Risk (VaR)

Let X denote the (one-period) **change in value** of a position over a fixed horizon. Fix a tail probability level $\alpha \in (0, 1)$.

Definition (VaR for a long position)

The loss event for a long position corresponds to a **large negative** X:

$$\mathrm{VaR}_L(\alpha) := -\inf\left\{x \in \mathbb{R} : \ \mathbb{P}(X \leq x) \geq \alpha\right\} \ = \ -q_\alpha(X).$$

Definition (VaR for a short position)

The loss event for a short position corresponds to a **large positive** X:

$$\mathrm{VaR}_S(\alpha) := \inf\left\{x \in \mathbb{R} : \ \mathbb{P}(X \geq x) \leq \alpha\right\} \ = \ q_{1-\alpha}(X).$$

Remark: VaR is a *tail quantile* of X, with the sign convention chosen so that VaR is positive under losses.

Expected Shortfall (ES)

Let X be the (one-period) **change in value** of a position over a fixed horizon. Fix a tail probability level $\alpha \in (0,1)$.

Definition (ES for a long position)

$$\mathrm{ES}_L(\alpha) := -\,\mathbb{E}\big[X \,\big|\, X \le -\mathrm{VaR}_L(\alpha)\big].$$

Definition (ES for a short position)

$$\mathrm{ES}_S(\alpha) := \mathbb{E}\big[X \,\big|\, X \ge \mathrm{VaR}_S(\alpha)\big].$$

Remark: ES measures the **average loss beyond VaR** (a tail conditional expectation).

Coherent risk measure

A risk measure is a mapping $\rho(\cdot)$ from random returns to $\mathbb{R}$.

Definition (Coherence)

A risk measure ρ is called **coherent** if it satisfies:

- **Monotonicity:** if $X \leq Y$ a.s., then $\rho(X) \geq \rho(Y)$.
- **Subadditivity:** $\rho(X + Y) \leq \rho(X) + \rho(Y)$.
- **Linearity:** for all $a, b \in \mathbb{R}$, we have $\rho(aX + b) = a\rho(X) - b$.

Remark: VaR is *not* coherent (can fail subadditivity), while ES is coherent.

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Seasonal ARIMA model (SARIMA)

Definition (ARIMA$(p, d, q) \times (P, D, Q)_s$)

x_t follows ARIMA$(p, d, q) \times (P, D, Q)_s$ if

$$\Phi_P(B^s)\Phi(B)\Delta_s^D\Delta^d x_t = \alpha + \Theta_Q(B^s)\Theta(B)\varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma_\varepsilon^2),$$

where $\Phi(B)$ and $\Theta(B)$ are non-seasonal AR/MA polynomials of orders p, q, and $\Phi_P(B^s)$ and $\Theta_Q(B^s)$ are seasonal AR/MA polynomials of orders P, Q.

For example:

- ARIMA$(1, 1, 1) \times (0, 1, 1)_{12}$:

$$(1 - \phi_1 B)(1 - B)(1 - B^{12})x_t = \alpha + (1 + \Theta_1 B^{12})(1 + \theta_1 B)\varepsilon_t.$$

- ARIMA$(2, 0, 0) \times (1, 0, 1)_4$:

$$(1 - \Phi_1 B^4)(1 - \phi_1 B - \phi_2 B^2)x_t = \alpha + (1 + \Theta_1 B^4)\varepsilon_t.$$