ST326 Week 3

Kaixin Liu[1]

[1]PhD Student in Statistics, LSE

Oct 17, 2025

Table of Contents

Table of Contents

White noise process

Definition (White noise process, WN)

If $\{x_t\}$ is a white noise process, then

- $\mathbb{E}(x_t) = \mu < \infty$,
- $s_\tau = 0$ for $\tau \neq 0$. $s_0 < \infty$ and is time independent,

shorthanded as $x_t \sim \text{WN}(\mu, \sigma^2)$ with mean μ and variance σ^2.

Examples:

- $x_t \overset{\text{i.i.d.}}{\sim} N(0, 1)$
- x_t's are independent but not identically distributed:
  - $x_t \sim \text{Exp}(\lambda)$ if t is odd;
  - $x_t \sim N(\lambda, \lambda^2)$ if t is even.

Table of Contents

AR and MA process

Definition (Autoregressive process of order p, AR(p))

A process $\{x_t\}$ is called an autoregressive process of order p, with shorthand $x_t \sim \text{AR}(p)$, if it can be written as

$$x_t = \alpha + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + \varepsilon_t,$$

where α and ϕ_j's are constants with $\phi_p \neq 0$, and $\varepsilon_t \sim \text{WN}(0, \sigma_\varepsilon^2)$.

Definition (Moving average process of order q, MA(q))

A process $\{x_t\}$ is called a moving average process of order q, with shorthand $x_t \sim \text{MA}(q)$, if it can be written as

$$x_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q},$$

where μ and θ_j's are constants with $\theta_q \neq 0$, and $\varepsilon_t \sim \text{WN}(0, \sigma_\varepsilon^2)$.

Identify AR and MA process

Definition (The partial autocorrelation function, PACF)

The PACF for a mean 0 stationary process $\{x_t\}$ is defined as

$$\begin{aligned}\pi(1) &= \mathrm{Corr}(x_2, x_1) \\ \pi(2) &= \mathrm{Corr}\big(x_3 - \widehat{\mathbb{E}}(x_3|x_2), x_1 - \widehat{\mathbb{E}}(x_1|x_2)\big) \\ \pi(3) &= \mathrm{Corr}\big(x_4 - \widehat{\mathbb{E}}(x_4|x_3, x_2), x_1 - \widehat{\mathbb{E}}(x_1|x_2, x_3)\big) \\ &\cdots\end{aligned}$$

where $\widehat{\mathbb{E}}$ is the linear operator fo finding the best linear representation (in terms of least squares) of y given $\mathbf{x}$, i.e., $\widehat{\mathbb{E}}(y|\mathbf{x}) = \alpha_0 + \boldsymbol{\alpha}_t^\top \mathbf{x}$ where $\alpha_0, \boldsymbol{\alpha}$ are the solutions to

$$\min_{\alpha_0, \boldsymbol{\alpha}} \mathbb{E}(y - \alpha_0 - \boldsymbol{\alpha}_t^\top \mathbf{x})^2.$$

Identify AR and MA process (Cont.)

Proposition

AR(p)	**MA(q)**
$\pi(j) = 0$ for $j > p$	$\rho(j) = 0$ for $j > q$

Table: Key identification fingerprints via ACF (ρ) and PACF (π).

- **AR(p):** PACF $\pi(j)$ *cuts off* at lag p; ACF $\rho(j)$ *tails off*.
- **MA(q):** ACF $\rho(j)$ *cuts off* at lag q; PACF $\pi(j)$ *tails off*.