# ST326 Week 5

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Oct 31, 2025

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# GARCH process

Volatility clustering of financial time series is more persistent than what an ARCH model can capture in general, in the sense that the theoretical autocorrelation of a squared ARCH($p$) process with small/moderate value of $p$ is often less than that for real data in practice.

**Definition (GARCH(p,q) process)**

The generalised ARCH model of order $p, q$, or GARCH($p, q$), is defined by

$$x_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 x_{t-1}^2 + \cdots + \alpha_p x_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2,$$

where $\{\varepsilon_t\} \sim IID(0,1)$ with $\alpha_p > 0$ and $\alpha_i \geq 0, \beta_j \geq 0$ for $i < p, j < q$, and $\sum_{i=1}^{\max(p,q)} (\alpha_i + \beta_i) < 1$.

# Fitting a GARCH model with Gaussian innovations

Assume Gaussian innovations $\varepsilon_t \sim$ i.i.d. $N(0,1)$. We then have

$$x_t \mid \mathcal{F}_{t-1} \sim N(0, \sigma_t).$$

Then, consider a standard decomposition of densities:

$$\begin{aligned} f(x_1, \ldots, x_T) &= f(x_T | x_{T-1}, \ldots, x_1) f(x_{T-1} | x_{T-2}, \ldots, x_1) \cdots \\ &= \Big( \prod_{t=p+1}^{T} \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\big(-\frac{x_t^2}{2\sigma_t^2}\big) \Big) \cdot f(x_p, \ldots, x_1). \end{aligned}$$

To maximise the joint density is to maximise

$$\prod_{t=p+1}^{T} \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\big(-\frac{x_t^2}{2\sigma_t^2}\big),$$

where values of $\sigma_t^2$ for $t = p - q + 1, \ldots, p$ are usually initialised as $\sigma_t^2 = x_t^2$.