ST326 Week 9

Kaixin Liu[1]

[1]PhD Student in Statistics, LSE

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LASSO

Consider again the original OLS problem, which is to solve

$$\widehat{\boldsymbol{\alpha}} = \arg\min_{\alpha} \|\mathbf{Y} - \mathbf{Z}\boldsymbol{\alpha}\|^2.$$

To restrict the magnitudes of $\boldsymbol{\alpha}$, we can solve

$$\min_{\alpha} \|\mathbf{Y} - \mathbf{Z}\boldsymbol{\alpha}\|^2, \qquad \text{subject to } \sum_{i=1}^{p} |\alpha_i| \leq c,$$

where $c > 0$. Using Lagrange multiplier, the above problem is equivalent to

$$\min_{\alpha} \left\{ \|\mathbf{Y} - \mathbf{Z}\boldsymbol{\alpha}\|^2 + \delta \sum_{i=1}^{p} |\alpha_i| \right\}$$

for some $\delta > 0$.

LASSO (Cont.)

To illustrate the variable selection ability of LASSO, consider $p = 2$, with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)^\top$. Assuming $\widehat{\boldsymbol{\alpha}} = (\mathbf{Z}^\top\mathbf{Z})^{-1}\mathbf{Z}^\top\mathbf{Y}$ exists, then $\|\mathbf{Y} - \mathbf{Z}\boldsymbol{\alpha}\|^2 = c$ for some constant $c > 0$ represents an ellipse on the α_1-α_2 plane, with center at $\widehat{\boldsymbol{\alpha}}$.

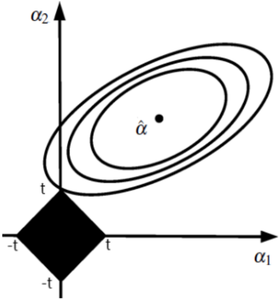


Figure: Left: For Lasso. Right: For ridge regression. The solutions are when then elliptical contours touch the diamond (Lasso) or the circle (ridge regression).

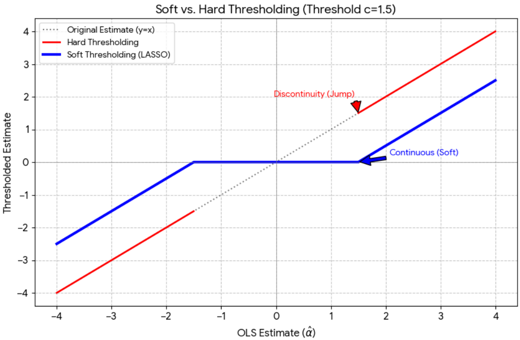
LASSO (Cont.)

If $\mathbf{Z}^\top \mathbf{Z} = n\mathbf{I}_p$, then we can show that the Lasso estimator is

$$\widetilde{\alpha}_j = \widehat{\alpha}_j \max\left\{0, 1 - \frac{\delta/2n}{|\widehat{\alpha}_j|}\right\} =: f_{\delta/2n}(\widehat{\alpha}_j),$$

where $f_c(x)$ is called a soft-thresholding function with parameter c, that

$$f_c(x) = x \max\left\{0, 1 - \frac{c}{|x|}\right\}.$$



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p-risky assets

Given a target return μ^*, the general problem we want to solve is

$$\min_{\mathbf{w}} \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} \qquad \text{s.t. } \mathbf{w}^\top \mathbf{1}_p = 1, \;\; \mu^* \leq \mathbf{w}^\top \boldsymbol{\mu} \tag{3.1}$$

Theorem (Two Fund Theorem)

The solution to (3.1) *is of the form*

$$\mathbf{w}_{opt} = (1-\alpha)\mathbf{w}_{mv} + \alpha \mathbf{w}_{mkt},$$

where

$$\mathbf{w}_{mv} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}_p}{\mathbf{1}_p^\top \boldsymbol{\Sigma}^{-1}\mathbf{1}_p}, \qquad \mathbf{w}_{mkt} = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1}\mathbf{1}_p}.$$

Determining α (p-risky assets, Cont.)

The portfolio $\mathbf{w}_{opt}$ must satisfy the return constraint

$$\mathbf{w}_{opt}^\top \boldsymbol{\mu} = \mu^*.$$

Substituting $\mathbf{w}_{opt} = (1-\alpha)\mathbf{w}_{mv} + \alpha \mathbf{w}_{mkt}$ gives

$$\mu^* = (1-\alpha)\,\mathbf{w}_{mv}^\top \boldsymbol{\mu} + \alpha\,\mathbf{w}_{mkt}^\top \boldsymbol{\mu}.$$

Solving for α yields

$$\alpha = \frac{\mu^* - \mathbf{w}_{mv}^\top \boldsymbol{\mu}}{\mathbf{w}_{mkt}^\top \boldsymbol{\mu} - \mathbf{w}_{mv}^\top \boldsymbol{\mu}}.$$

If $\mu^* \leq \mathbf{w}_{mv}^\top \boldsymbol{\mu}$, then the minimum-variance portfolio already satisfies the return requirement and $\alpha = 0$.

Economic Interpretation (p-risky assets, Cont.)

$$\mathbf{w}_{mv} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}_p}{\mathbf{1}_p^\top \boldsymbol{\Sigma}^{-1}\mathbf{1}_p}, \qquad \mathbf{w}_{mkt} = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1}\mathbf{1}_p}.$$

- Every efficient portfolio is a combination of only **two** funds:

$$\mathbf{w}_{mv} \qquad \text{and} \qquad \mathbf{w}_{mkt}.$$

  - $\mathbf{w}_{mv}$ minimizes risk regardless of the target return.
  - $\mathbf{w}_{mkt}$ maximizes the Sharpe ratio (tangent portfolio in CAPM).
- Changing the target return μ^* only changes the mixing weight α.
- When μ is proportional to $\mathbf{1}_p$, $\mathbf{w}_{mkt} = \mathbf{w}_{mv}$ (assets have identical expected returns), and the return constraint becomes infeasible for $\mu^* > \mathbf{w}_{mv}^\top \boldsymbol{\mu}$.

p-risky assets + risk-free asset

Let r_f be the return of a risk-free asset. A portfolio now consists of $(w_0, \mathbf{w}^\top)^\top$ where w_0 is the weight on the risk-free asset and $\mathbf{w} \in \mathbf{R}^p$ allocates to the risky assets.

Since weights must sum to one,

$$w_0 + \mathbf{w}^\top \mathbf{1}_p = 1.$$

The efficient frontier problem becomes

$$\min_{w_0, \mathbf{w}} \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} \quad \text{s.t.} \quad w_0 + \mathbf{w}^\top \mathbf{1}_p = 1, \quad \mu^* \leq w_0 r_f + \mathbf{w}^\top \boldsymbol{\mu}.$$

Eliminating w_0 yields the equivalent problem

$$\min_{\mathbf{w}} \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} \quad \text{s.t.} \quad \mu^* - r_f \leq \mathbf{w}^\top (\boldsymbol{\mu} - r_f \mathbf{1}_p).$$

Solution (p-risky assets + risk-free asset, Cont.)

If $r_f \geq \mu^*$, the optimal solution is trivially $\mathbf{w} = 0$: invest everything in the risk-free asset.

If $r_f < \mu^*$, the constraint binds, and solving the Lagrangian yields

$$\mathbf{w}_{opt} = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f \mathbf{1}_p)}{(\boldsymbol{\mu} - r_f \mathbf{1}_p)^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f \mathbf{1}_p)}(\mu^* - r_f) = \mathbf{w}^0_{mkt}\,(\mu^* - r_f),$$

where the market portfolio of risky assets is

$$\mathbf{w}^0_{mkt} = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f \mathbf{1}_p)}{\mathbf{1}_p^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f \mathbf{1}_p)}.$$

Note that $\mathbf{w}^0_{mkt}$ is independent of μ^* and satisfies $\mathbf{1}_p^\top \mathbf{w}^0_{mkt} = 1$.

One-Fund Theorem

Theorem (One-Fund Theorem)

With a risk-free asset available, every efficient portfolio can be expressed as a combination of:

(i) the risk-free asset, *(ii) the market portfolio* $\mathbf{w}^0_{mkt}$.

That is,

$$\textit{Efficient portfolio} = w_0 r_f \; + \; (1 - w_0)\mathbf{w}^0_{mkt}.$$

This differs from the two-fund theorem: **only one risky fund is needed**.