ST418 Week 2

Kaixin Liu[1]

[1]PhD Student in Statistics, LSE

Jan 29, 2026

Table of Contents

Table of Contents

About me

Education:

- Second year PhD student.
- Supervision under Prof. Clifford Lam and Dr. Yunxiao Chen.
- Research interests in: High-dimensional time series analysis; Latent factor model with vector, matrix and tensor time series.
- Obtained BEcon in major of Economic Statistics from Beihang University (formerly known as Beijing University of Aeronautics and Astronautics) in 2023, and MSc in major of Statistics (Financial Statistics) from LSE in 2024.

How to reach out:

- Email Address: K.Liu31@lse.ac.uk
- Office: Columbia House 5.02
- Office Hour: TBD
- Homepage: https://www.lse.ac.uk/people/kaixin-liu

Table of Contents

Autocovariance and autocorrelation (general case)

For a general time series $\{x_t\}$, define the mean function

$$\mu_t = \mathbb{E}(x_t),$$

and the **autocovariance function** (two-time indices)

$$\gamma(s,t) = \mathrm{Cov}(x_s, x_t) = \mathbb{E}\big[(x_s - \mu_s)(x_t - \mu_t)\big].$$

The **autocorrelation function** is

$$\rho(s,t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}} \in [-1,1].$$

Interpretation: $\rho(s,t)$ measures linear predictability of x_t using only x_s.

Table of Contents

Strict (strong) stationarity

Definition (Strict / strong stationarity)

$\{x_t\}$ is **strictly stationary** if for any $m > 0$, any time points $t_1, \ldots, t_m$, and any shift $h \in \mathbb{Z}$,

$$\mathbb{P}(x_{t_1} \leq c_1, \ldots, x_{t_m} \leq c_m) = \mathbb{P}(x_{t_1+h} \leq c_1, \ldots, x_{t_m+h} \leq c_m)$$

for all real constants $c_1, \ldots, c_m$.

Immediate implications:

- $(m = 1)$ all x_t share the same marginal distribution $\Rightarrow$ if μ_t exists then $\mu_t \equiv \mu$.
- $(m = 2)$ if second moments exist, then $\gamma(s, t) = \gamma(s + h, t + h)$ (depends only on lag).

Table of Contents

Weak (second-order) stationarity

Definition (Weak / second-order stationarity)

$\{x_t\}$ is **weakly stationary** if

- $\mathbb{E}(x_t) = \mu < \infty$ (constant over t), and
- $\gamma(s, t)$ depends on s, t only through $|s - t|$, with $\mathrm{Var}(x_t) = \gamma(t, t) < \infty$.

Notation simplification under stationarity:

$$\gamma(h) = \mathrm{Cov}(x_{t+h}, x_t).$$

For discrete/equally spaced series, define the **ACVS**

$$s_\tau = \mathrm{Cov}(x_{t+\tau}, x_t), \quad \tau \in \mathbb{Z}.$$

ACF for a stationary time series

For a weakly stationary series, the **ACF** is

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}.$$

For integer lags (discrete time),

$$\rho_\tau = \frac{s_\tau}{s_0}, \qquad s_0 = \gamma(0) = \mathrm{Var}(x_t).$$

So: ACVS = covariance by lag; ACF = ACVS normalised by variance.

When weak stationarity implies strict stationarity

Definition (Gaussian process)

$\{x_t\}$ is a **Gaussian process** if for any n and any time points $t_1, \ldots, t_n$, the vector $(x_{t_1}, \ldots, x_{t_n})$ is multivariate normal with finite means and variances.

Key fact

A multivariate normal distribution is fully characterised by its mean vector and covariance matrix.

Theorem (Weakly stationary Gaussian $\Rightarrow$ strictly stationary)

If $\{x_t\}$ is a Gaussian process and weakly stationary, then $\{x_t\}$ is strictly stationary.

Table of Contents

Positive semidefiniteness of the ACVS

Definition (Positive semidefinite sequence)

A sequence $\{s_\tau\}$ is **positive semidefinite** if for any time points $t_1, \ldots, t_n$ and any real numbers $a_1, \ldots, a_n$ (not all zero),

$$\sum_{i=1}^{n} \sum_{j=1}^{n} s_{t_i - t_j} \, a_i a_j \geq 0.$$

Why it must hold for stationary time series

If $\{s_\tau\}$ is the ACVS of a stationary $\{x_t\}$, then

$$0 \leq \mathrm{Var}\left(\sum_{i=1}^{n} a_i x_{t_i} \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} s_{t_i - t_j} \, a_i a_j,$$

hence $\{s_\tau\}$ is positive semidefinite.

Conversely: any positive semidefinite sequence can be an ACVS of some stationary process.