ST418 Week 4

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Table of Contents

Table of Contents

ARMA process

Definition (ARMA(p,q) process)

A process $\{x_t\}$ is called an ARMA(p, q) process, if

$$x_t = \alpha + \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q},$$

where the ϕ_j's and θ_j'sare all constants with $\phi_p, \theta_q \neq 0$, and $\varepsilon_t \sim$ WN$(0, \sigma_\varepsilon^2)$, shorthanded as $x_t \sim$ ARMA(p, q).

Definition (AR and MA operators)

Define the autoregressive and moving average operator, respectively as

$$\Phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p,$$
$$\Theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q.$$

Therefore, the cumbersome representation of ARMA(p, q) model can be expressed as $\Phi(B)x_t = \alpha + \Theta(B)\varepsilon_t$.

AR and MA characteristic polynomial

Definition (AR and MA characteristic polynomial)

Define the AR and MA characteristic polynomial, respectively as

$$\Phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p,$$
$$\Theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q.$$

Remark (common factors and over-parameterisation):
Writing $\Phi(B)x_t = \Theta(B)\varepsilon_t$ makes it easy to spot common factors in $\Phi(\cdot)$ and $\Theta(\cdot)$ (which may indicate over-parameterisation) and when they can be canceled (when the corresponding roots lie outside the unit circle).

Table of Contents

Stationarity and Invertibility of ARMA process

Theorem (Stationarity of ARMA(p,q))

If $x_t \sim \text{ARMA}(p,q)$ *s.t.* $\Phi(B)x_t = \alpha + \Theta(B)\varepsilon_t$, *then* $\{x_t\}$ *is stationary if and only if all the roots of* $\Phi(z) = 0$ *lie outside the unit circle.*

Definition (Invertibility)

If $\{x_t\}$ can be written as an $\text{AR}(\infty)$ process, i.e.,

$$\varepsilon_t = \sum_{j\geq 0} \psi_j x_{t-j},$$

where $\varepsilon_t \sim \text{WN}(0, \sigma_\varepsilon^2)$, then we say $\{x_t\}$ is invertible.

Theorem (Invertibility of ARMA(p,q))

If $x_t \sim \text{ARMA}(p,q)$ *s.t.* $\Phi(B)x_t = \alpha + \Theta(B)\varepsilon_t$, *then* $\{x_t\}$ *is invertible if and only if all the roots of* $\Theta(z) = 0$ *lie outside the unit circle.*

A brief summary

Properties of AR, MA, ARMA processes

	AR(p)	**MA(q)**	**ARMA(p, q)**
Stationarity	Roots of $\Phi(z)$ outside $\lvert z\rvert \leq 1$	Always stationary	Roots of $\Phi(z)$ outside $\lvert z\rvert \leq 1$
Invertibility	Always invertible	Roots of $\Theta(z)$ outside $\lvert z\rvert \leq 1$	Roots of $\Theta(z)$ outside $\lvert z\rvert \leq 1$
ACF $\rho(j)$	Tails off	$\rho(j) = 0$ for $j > q$	Tails off
PACF $\pi(j)$	$\pi(j) = 0$ for $j > p$	Tails off	Tails off

Table: Stationarity, invertibility, and identification fingerprints via ACF (ρ) and PACF (π). "Tails of" typically means exponential (possibly damped).