ST326 Week 4

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ARMA process

Definition (ARMA(p,q) process)

A process $\{x_t\}$ is called an ARMA(p, q) process, if

$$x_t = \alpha + \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q},$$

where the ϕ_j's and θ_j'sare all constants with $\phi_p, \theta_q \neq 0$, and $\varepsilon_t \sim \text{WN}(0, \sigma_\varepsilon^2)$, shorthanded as $x_t \sim \text{ARMA}(p, q)$.

Definition (AR and MA operators)

Define the autoregressive and moving average operator, respectively as

$$\Phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p,$$
$$\Theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q.$$

Therefore, the cumbersome representation of ARMA(p, q) model can be expressed as $\Phi(B)x_t = \alpha + \Theta(B)\varepsilon_t$.

AR and MA characteristic polynomial

Definition (AR and MA characteristic polynomial)

Define the AR and MA characteristic polynomial, respectively as

$$\Phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p,$$
$$\Theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q.$$

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Stationarity and Invertibility of ARMA process

Theorem (Stationarity of ARMA(p,q))

If $x_t \sim \text{ARMA}(p, q)$ *s.t.* $\Phi(B)x_t = \alpha + \Theta(B)\varepsilon_t$, *then* $\{x_t\}$ *is stationary if and only if all the roots of* $\Phi(z) = 0$ *lie outside the unit circle.*

Definition (Invertibility)

If $\{x_t\}$ can be written as an AR(∞) process, i.e.,

$$\varepsilon_t = \sum_{j \geq 0} \psi_j x_{t-j},$$

where $\varepsilon_t \sim \text{WN}(0, \sigma_\varepsilon^2)$, then we say $\{x_t\}$ is invertible.

Theorem (Invertibility of ARMA(p,q))

If $x_t \sim \text{ARMA}(p, q)$ *s.t.* $\Phi(B)x_t = \alpha + \Theta(B)\varepsilon_t$, *then* $\{x_t\}$ *is invertible if and only if all the roots of* $\Theta(z) = 0$ *lie outside the unit circle.*

A brief summary

Properties of AR, MA, ARMA processes

	AR(p)	**MA(q)**	**ARMA(p,q)**
Stationarity	Roots of $\Phi(z)$ outside $\lvert z\rvert \leq 1$	Always stationary	Roots of $\Phi(z)$ outside $\lvert z\rvert \leq 1$
Invertibility	Always invertible	Roots of $\Theta(z)$ outside $\lvert z\rvert \leq 1$	Roots of $\Theta(z)$ outside $\lvert z\rvert \leq 1$
ACF $\rho(j)$	Tails off	$\rho(j) = 0$ for $j > q$	Tails off
PACF $\pi(j)$	$\pi(j) = 0$ for $j > p$	Tails off	Tails off

Table: Stationarity, invertibility, and identification fingerprints via ACF (ρ) and PACF (π). "Tails of" typically means exponential (possibly damped).

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Properties of volatility for financial time series

Volatility is simply the conditional standard deviation,

$$\sigma_t = \sqrt{\mathrm{Var}(r_t | \mathcal{F}_{t-1})},$$

where r_t is the underlying asset log return, and $\mathcal{F}_{t-1}$ denotes past information until $t - 1$.

- **Volatility clustering**: Volatility of asset returns tend to be high for a certain period of time, and low for other periods.
- **Continuity**: Volatility evolves continuously.
- **Boundedness**: Volatility varies within some fixed range. Hence it is usually stationary.
- **Leverage**: Volatility tends to react differently to big price increase or a big price drop.

ARCH process

Definition (ARCH(p) process)

The autoregressive conditional heteroscedastic model of order p, or ARCH(p), is defined by

$$x_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 x_{t-1}^2 + \cdots + \alpha_p x_{t-p}^2,$$

where $\{\varepsilon_t\} \sim IID(0,1)$ with $\alpha_p > 0$ and $\alpha_i \geq 0$ for $i < p$.

Properties

If $x_t \sim \text{ARCH}(p)$, then:

- **Uncorrelatedness**: $\{x_t\}$ is a zero mean white noise.
- **Clustering of large and small values.**
- **The x_t's are not independent**: $\text{Cov}(x_t^2, x_{t-1}^2) = \alpha_1 \text{Var}(x_t^2) > 0$.