ST304 Week 11

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Return characteristics

- **Prices vs returns:** prices P_t are usually non-stationary, while log-returns $X_t = \log P_t - \log P_{t-1}$ are often close to serially uncorrelated.
- **Volatility clustering:** large $|X_t|$ or X_t^2 tend to be followed by large values again.
- **Heavy tails:** extreme returns occur more often than under a Gaussian model.
- **Motivation:** ARMA with independent innovations gives constant variance, so ARCH/GARCH model volatility directly.

Definition of ARCH and GARCH processes

Definition (ARCH(q) and GARCH(p, q))

Let $\widetilde{X}_{t-1} := (X_s)_{s<t}$. For $q \in \mathbb{N}_0$, an ARCH(q) process satisfies

$$X_t = \sigma_t \varepsilon_t, \qquad \sigma_t^2 = \alpha_0 + \sum_{k=1}^{q} \alpha_k X_{t-k}^2,$$

for all t, where $\alpha_0, \alpha_1, \ldots, \alpha_q \geq 0$.
More generally, for $p, q \in \mathbb{N}_0$, a GARCH(p, q) process satisfies

$$X_t = \sigma_t \varepsilon_t, \qquad \sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 + \sum_{k=1}^{q} \alpha_k X_{t-k}^2,$$

where also $\beta_1, \ldots, \beta_p \geq 0$.

Here ε_t are i.i.d. with mean 0, variance 1, and independent of $\widetilde{X}_{t-1}$.

Some properties

Since σ_t^2 is measurable with respect to the past and independent of ε_t,

$$E(X_t \mid \widetilde{X}_{t-1}) = 0, \qquad E(X_t^2 \mid \widetilde{X}_{t-1}) = \sigma_t^2,$$

hence

$$\mathrm{Var}(X_t \mid \widetilde{X}_{t-1}) = \sigma_t^2.$$

- **ARCH terms** α_k**:** immediate effect of past shocks X_{t-k}^2.
- **GARCH terms** β_j**:** persistence through past volatilities σ_{t-j}^2.
- **Symmetry:** X_{t-k} and $-X_{t-k}$ have the same effect on σ_t^2.

Proposition 4.2

Suppose (X_t) is a weakly stationary GARCH(p, q) process.

- **(a) White noise:** $E(X_t) = 0$ and $E(X_t X_{t-h}) = 0$ for $h \geq 1$, with

  $$E(X_t^2) = \frac{\alpha_0}{1 - \sum_{j=1}^{p} \beta_j - \sum_{k=1}^{q} \alpha_k}.$$

  If $\alpha_0 > 0$, then $\sum_{j=1}^{p} \beta_j + \sum_{k=1}^{q} \alpha_k < 1$.
- **(b) Kurtosis:** $\kappa(X_t) \geq \kappa(\varepsilon_t)$. In particular, if $\varepsilon_t \sim N(0, 1)$, then $\kappa(X_t) \geq 3$.
- **(c) Squared returns:** if $E(X_t^4) < \infty$ and $m := \max(p, q)$, then (X_t^2) admits the ARMA(m, p) representation

  $$\eta_t := \sigma_t^2(\varepsilon_t^2 - 1),$$

  $$X_t^2 = \alpha_0 + \sum_{k=1}^{m} (\alpha_k + \beta_k) X_{t-k}^2 + \eta_t - \sum_{j=1}^{p} \beta_j \eta_{t-j}.$$

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Least squares and MLE

Least squares for ARCH(q). Since

$$X_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2 + \eta_t, \qquad \eta_t := \sigma_t^2(\varepsilon_t^2 - 1),$$

estimate $\alpha = (\alpha_0, \ldots, \alpha_q)^\top$ by regressing X_t^2 on $1, X_{t-1}^2, \ldots, X_{t-q}^2$:

$$\widehat{\boldsymbol{\alpha}} = (Z^\top Z)^{-1} Z^\top Y.$$

Gaussian MLE for GARCH(p, q). If $\varepsilon_t \overset{\text{i.i.d.}}{\sim} N(0,1)$, then $X_t \mid \widetilde{X}_{t-1} \sim N(0, \sigma_t^2)$ and

$$L_n(\boldsymbol{\theta}) = \prod_{t=1}^{n} (2\pi\sigma_t^2)^{-1/2} \exp\left(-\frac{X_t^2}{2\sigma_t^2}\right),$$

where σ_t^2 is computed recursively and $\boldsymbol{\theta} = (\alpha_0, \ldots, \alpha_q, \beta_1, \ldots, \beta_p)^\top$.

Forecasting

Returns. For every $h \geq 1$,

$$E(X_{n+h} \mid X_n, \ldots, X_1) = 0,$$

so GARCH models do not make future returns predictable in mean.
Volatility. Forecasting focuses on

$$E(X_{n+h}^2 \mid X_n, \ldots, X_1) = E(\sigma_{n+h}^2 \mid X_n, \ldots, X_1).$$

By Proposition 4.2(c), one may use the ARMA representation of (X_t^2).
For GARCH$(1,1)$,

$$\widehat{\sigma}_{n+1|n}^2 = \alpha_0 + \alpha_1 X_n^2 + \beta_1 \sigma_n^2, \qquad \widehat{\sigma}_{n+h|n}^2 = \alpha_0 + (\alpha_1 + \beta_1)\widehat{\sigma}_{n+h-1|n}^2$$

for $h \geq 2$. If $\alpha_1 + \beta_1 < 1$, then

$$\widehat{\sigma}_{n+h|n}^2 \to \frac{\alpha_0}{1 - \alpha_1 - \beta_1}.$$

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AR-GARCH, EGARCH and stochastic volatility

- **AR-GARCH:**

$$Y_t = \phi Y_{t-1} + X_t, \qquad X_t = \sigma_t \varepsilon_t.$$

- **EGARCH:**

$$\log \sigma_t = \alpha + \beta \log \sigma_{t-1} + \theta \varepsilon_{t-1} + \gamma \big(|\varepsilon_{t-1}| - E|\varepsilon_{t-1}| \big).$$

  This allows asymmetric responses and removes positivity constraints on the volatility recursion.

- **Stochastic volatility:**

$$X_t = G(h_t)\varepsilon_t, \qquad h_t = \sum_{j=1}^{p} \phi_j h_{t-j} + e_t.$$

  Here (h_t) is latent and (e_t) is independent of (ε_t).