ST304 Week 8

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Mar 12, 2026

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Asymptotic normality of sample ACF and PACF

AR(p): PACF cutoff

For a causal AR(p) process, $\alpha(h) = 0$ for all $h > p$, and for each fixed $h > p$,

$$\sqrt{n}\,\widehat{\alpha}_n(h) \xrightarrow{d} N(0, 1).$$

MA(q): ACF cutoff

For an MA(q) process, $\rho(h) = 0$ for all $h > q$, and for each fixed $h > q$,

$$\sqrt{n}\,\widehat{\rho}_n(h) \xrightarrow{d} N\Big(0,\ 1 + 2\rho(1)^2 + \cdots + 2\rho(q)^2\Big).$$

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Yule–Walker estimators

Yule–Walker equations for AR(p)

For a causal AR(p) process, the ACVF satisfies

$$\gamma(h) = \sum_{j=1}^{p} \phi_j \gamma(h-j), \quad h = 1, \ldots, p, \qquad \gamma(0) = \sum_{j=1}^{p} \phi_j \gamma(j) + \sigma^2.$$

Method-of-moments estimators

Replacing $\gamma(\cdot)$ by $\widehat{\gamma}_n(\cdot)$ gives

$$\widehat{\boldsymbol{\phi}} = \widehat{\boldsymbol{\Gamma}}^{-1} \widehat{\boldsymbol{\gamma}}, \qquad \widehat{\sigma}^2 = \widehat{\gamma}_n(0) - \widehat{\boldsymbol{\phi}}^\top \widehat{\boldsymbol{\gamma}},$$

where $\widehat{\boldsymbol{\gamma}} = (\widehat{\gamma}_n(1), \ldots, \widehat{\gamma}_n(p))^\top$ and $\widehat{\Gamma}_{j,k} = \widehat{\gamma}_n(j-k)$. Also, $\widehat{\phi}_p = \widehat{\alpha}_n(p)$ (sample PACF at lag p).

Yule–Walker estimators (cont.)

Theorem (causal AR(p))

If (X_t) is a causal AR(p) process built from i.i.d. white noise, then

$$\sqrt{n}\,(\widehat{\boldsymbol{\phi}} - \boldsymbol{\phi}) \xrightarrow{d} N_p(0, \sigma^2 \boldsymbol{\Gamma}^{-1}), \qquad \widehat{\sigma}^2 \xrightarrow{p} \sigma^2,$$

where $\boldsymbol{\Gamma} \in \mathbb{R}^{p \times p}$ has entries $\Gamma_{j,k} = \gamma(j-k)$ for $j, k \in \{1, \ldots, p\}$.

Example (special case $p = 1$). If $|\phi_1| < 1$ and $\widehat{\phi}_1 = \widehat{\rho}_n(1)$, then

$$\sqrt{n}\,(\widehat{\phi}_1 - \phi_1) \xrightarrow{d} N(0, 1 - \phi_1^2), \quad \mathrm{CI}_{95\%}:\ \widehat{\phi}_1 \pm 1.96\sqrt{\frac{1 - \widehat{\phi}_1^2}{n}}.$$

For MA(1), solving $\widehat{\rho}_n(1) = \widehat{\theta}/(1 + \widehat{\theta}^2)$ requires $|\widehat{\rho}_n(1)| \le 1/2$ for a real-valued $\widehat{\theta}$.