

ST326 Week 4

Kaixin Liu[1]

[1]PhD Student in Statistics, LSE

Oct 23, 2026

Table of Contents

1. Stationarity and white noise
2. Linear time series models

Table of Contents

1 **Stationarity and white noise**

2 Linear time series models

Strong and weak stationarity

Definition (Strong stationarity)

For every $m \geq 1$, all $t_1, \ldots, t_m \in \mathbb{Z}$ and every shift $h \in \mathbb{Z}$,

$$(X_{t_1}, \ldots, X_{t_m}) \stackrel{d}{=} (X_{t_1+h}, \ldots, X_{t_m+h}).$$

Here $\stackrel{d}{=}$ denotes equality of joint distributions.

Definition (Weak stationarity)

For every $t, k \in \mathbb{Z}$, the second moment is finite and

$$\mathbb{E}(X_t) = \mu, \qquad \mathrm{Cov}(X_t, X_{t+k}) = \gamma(k).$$

The mean and covariance at a fixed lag do not depend on t.

Strong stationarity with finite second moments implies weak stationarity; the converse need not hold.

White noise

Definition ($\mathrm{WN}(\mu, \sigma^2)$)

For all $t \in \mathbb{Z}$ and nonzero integer lags k,

$$\mathbb{E}(X_t) = \mu, \qquad \mathrm{Var}(X_t) = \sigma^2 < \infty, \qquad \mathrm{Cov}(X_t, X_{t+k}) = 0.$$

White noise has a constant mean and variance, and is serially uncorrelated.

We use $\varepsilon_t \sim \mathrm{WN}(0, \sigma^2)$ for shocks. White noise need not be independent or Normal; IID observations with finite variance are white noise.

Table of Contents

1 Stationarity and white noise

2 **Linear time series models**

Autoregressive model: AR(p)

Throughout, $\varepsilon_t \sim \mathrm{WN}(0, \sigma^2)$ and coefficients are constant.

Definition (AR(p))

$$X_t = c + b_1 X_{t-1} + \cdots + b_p X_{t-p} + \varepsilon_t.$$

The order p counts lagged observations; c is the intercept and b_j weights X_{t-j}.

For AR(1), $X_t = c + bX_{t-1} + \varepsilon_t$: the previous value feeds into the current value, together with a new shock. AR models are not automatically stationary; c is not generally the mean.

Moving average model: MA(q)

Definition (MA(q))

$$X_t = \mu + \varepsilon_t + a_1\varepsilon_{t-1} + \cdots + a_q\varepsilon_{t-q}, \qquad \varepsilon_t \sim \mathrm{WN}(0, \sigma^2).$$

The order q counts lagged shocks; μ is the mean and a_j weights the shock j periods ago.

For MA(1), $X_t = \mu + \varepsilon_t + a\varepsilon_{t-1}$: a shock enters the current and next observation directly. The model combines unobserved shocks rather than averaging observed prices.

Autoregressive moving average model: ARMA(p, q)

Definition (ARMA(p, q))

$$X_t = c + \sum_{j=1}^{p} b_j X_{t-j} + \varepsilon_t + \sum_{j=1}^{q} a_j \varepsilon_{t-j}.$$

Here $\varepsilon_t \sim \mathrm{WN}(0, \sigma^2)$, p counts lagged observations and q counts lagged shocks.

ARMA(1,1) is $X_t = c + bX_{t-1} + \varepsilon_t + a\varepsilon_{t-1}$: it combines past values and past shocks. Setting $q = 0$ gives AR(p); setting $p = 0$ gives MA(q), with $c = \mu$.

Integrated model: ARIMA(p, d, q)

Definition (ARIMA(p, d, q))

Write $\nabla X_t = X_t - X_{t-1}$ and let ∇^d mean differencing d times.
$X_t \sim \mathrm{ARIMA}(p, d, q)$ if $Y_t = \nabla^d X_t$ is stationary ARMA(p, q):

$$Y_t = c + \sum_{j=1}^{p} b_j Y_{t-j} + \varepsilon_t + \sum_{j=1}^{q} a_j \varepsilon_{t-j}.$$

Here $\varepsilon_t \sim \mathrm{WN}(0, \sigma^2)$; $d \geq 0$ is the differencing order.

$d = 0$ gives ARMA. If $X_t = \log P_t$, then $\nabla X_t = r_t$: one difference converts log prices into log returns, so we model changes rather than levels.