

ST326 Week 6

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Backshift notation

Definition (Backshift operator)

The operator B shifts a series back by one period:

$$BX_t = X_{t-1}, \qquad B^j X_t = X_{t-j}.$$

Definition (AR and MA polynomials)

$$b(z) = 1 - \sum_{j=1}^{p} b_j z^j, \qquad a(z) = 1 + \sum_{j=1}^{q} a_j z^j.$$

Replacing z by B gives the operators $b(B)$ and $a(B)$.

These operators collect lagged values or shocks into a compact expression. Root conditions below assume no common AR and MA factors.

AR, MA and ARMA models

Throughout, $\varepsilon_t \sim \mathrm{WN}(0, \sigma^2)$.

Definition (AR(p))

$$X_t = c + \sum_{j=1}^{p} b_j X_{t-j} + \varepsilon_t, \quad b(B)X_t = c + \varepsilon_t.$$

Definition (MA(q))

$$X_t = \mu + \varepsilon_t + \sum_{j=1}^{q} a_j \varepsilon_{t-j}, \quad X_t = \mu + a(B)\varepsilon_t.$$

Definition (ARMA(p, q))

$$X_t = c + \sum_{j=1}^{p} b_j X_{t-j} + \varepsilon_t + \sum_{j=1}^{q} a_j \varepsilon_{t-j}, \quad b(B)X_t = c + a(B)\varepsilon_t.$$

Stationarity and causality

Definition (Weak stationarity)

Finite second moments, a constant mean μ, and covariance depending only on the lag:

$$\mathbb{E}(X_t) = \mu, \qquad \mathrm{Cov}(X_t, X_{t+k}) = \gamma(k).$$

Definition (Causality)

A causal ARMA solution has the representation

$$X_t - \mu = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \qquad \sum_{j=0}^{\infty} |\psi_j| < \infty.$$

Only present and past shocks enter X_t.

A stationary causal solution exists iff every root of $b(z) = 0$ has $|z| > 1$. This is the stationarity criterion used in the lecture notes.

Invertibility

Let $Y_t = X_t - \mu$ denote the centred stationary series.

Definition (Invertibility)

The shocks can be recovered from current and past observations:

$$\varepsilon_t = \sum_{j=0}^{\infty} \pi_j Y_{t-j}, \qquad \sum_{j=0}^{\infty} |\pi_j| < \infty.$$

Root criterion

Every root of $a(z) = 0$ must have $|z| > 1$. Then

$$\varepsilon_t = a(B)^{-1} b(B) Y_t.$$

Causality expresses observations through shocks; invertibility expresses shocks through observations.

Model properties: comparison

	AR(p)	MA(q)	ARMA(p, q)
Stationarity*	AR roots $\lvert z\rvert > 1$	Always	AR roots $\lvert z\rvert > 1$
Causality	AR roots $\lvert z\rvert > 1$	Always	AR roots $\lvert z\rvert > 1$
Invertibility	Always	MA roots $\lvert z\rvert > 1$	MA roots $\lvert z\rvert > 1$

AR roots solve $b(z) = 0$; MA roots solve $a(z) = 0$. *Stationarity refers to the causal solution used in the notes. In general, stationarity and causality are distinct properties.

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Least squares and maximum likelihood

Fix (p, q) and write $\theta = (c, b_1, \ldots, b_p, a_1, \ldots, a_q)$.

Least squares (LSE)

Minimise squared residuals after initialisation:

$$\hat{\theta}_{\mathrm{LS}} = \arg\min_{\theta} \sum_{t=\max(p,q)+1}^{T} \hat{\varepsilon}_t(\theta)^2.$$

AR fitting is linear regression. ARMA residuals depend recursively on the parameters, so numerical optimisation is needed.

Maximum likelihood (MLE)

Under IID Gaussian innovations, maximise the joint likelihood of $X_1, \ldots, X_T$ over θ and σ^2.

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Residual analysis and the Ljung–Box test

Residuals $\hat{\varepsilon}_t = X_t - \hat{X}_t$ are estimated one-step prediction errors. An adequate mean model should leave no serial correlation.

- Check residuals fluctuate around zero without a systematic pattern, and inspect their sample ACF.
- A Normal Q–Q plot assesses tails when Gaussian errors are assumed.

Ljung–Box test

H_0: residual autocorrelations at lags $1, \ldots, m$ are zero.

$$Q_m = T(T+2)\sum_{j=1}^{m} \frac{\hat{\rho}_{\hat{\varepsilon}}(j)^2}{T-j} \overset{a}{\sim} \chi^2_{m-p-q}, \qquad m > p+q.$$

Reject for large Q_m. The degrees-of-freedom adjustment accounts for fitting the AR and MA coefficients.

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EMA and MACD

Let P_t be the price and $\lambda_k = (k-1)/(k+1)$.

Definition (EMA)

Older prices receive geometrically decreasing weights:

$$\mathrm{EMA}_k(t) = (1-\lambda_k)P_t + \lambda_k \mathrm{EMA}_k(t-1).$$

Definition (MACD)

For short and long periods $s < l$,

$$\mathrm{MACD}(t) = \mathrm{EMA}_s(t) - \mathrm{EMA}_l(t).$$

A signal line is an EMA of MACD, commonly over 9 periods.

Larger k gives smoother EMA values. Positive MACD means the short average exceeds the long average. Crossovers can lag price changes.

Relative Strength Index (RSI)

Let $G_t = \max(P_t - P_{t-1}, 0)$ and $L_t = \max(P_{t-1} - P_t, 0)$ be the current gain and loss.

Definition (RSI)

Smooth gains and losses over k periods:

$$\mathrm{avGain}_k(t) = \frac{1}{k} G_t + \frac{k-1}{k} \mathrm{avGain}_k(t-1),$$

$$\mathrm{avLoss}_k(t) = \frac{1}{k} L_t + \frac{k-1}{k} \mathrm{avLoss}_k(t-1),$$

$$\mathrm{RSI}_k(t) = 100 \frac{\mathrm{avGain}_k(t)}{\mathrm{avGain}_k(t) + \mathrm{avLoss}_k(t)}.$$

For a positive denominator, RSI lies in $[0, 100]$.

Usually $k = 14$. RSI above 70 is called overbought; below 30, oversold. Neither guarantees a price reversal.