ST418 Week 10

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Multivariate time series and weak stationarity

Let $\mathbf{x}_t = (x_{t1}, \ldots, x_{tN})^\top$ where $t \in \mathcal{I}$ so that $\{\mathbf{x}_t\}$ is an N-dimensional time series with mean vector $\boldsymbol{\mu}_t = \mathbb{E}(\mathbf{x}_t)$ and covariance matrix

$$\mathrm{Cov}(\mathbf{x}_s, \mathbf{x}_t) = \mathbb{E}\big[(\mathbf{x}_s - \boldsymbol{\mu}_s)(\mathbf{x}_t - \boldsymbol{\mu}_t)^\top\big] = \big(\gamma_{ij}(s,t)\big)_{i,j=1}^{N}.$$

Definition (Weak stationarity)

$\{\mathbf{x}_t\}$ is weakly stationary if

$$\mathbb{E}(\mathbf{x}_t) = \boldsymbol{\mu} \quad \text{and} \quad \boldsymbol{\Gamma}_\tau = \mathrm{Cov}(\mathbf{x}_t, \mathbf{x}_{t+\tau})$$

depend only on the lag τ, with finite entries. Write

$$\boldsymbol{\Gamma}_\tau = (s_{ij,\tau}), \qquad s_{ij,\tau} = \mathrm{Cov}(x_{t,i}, x_{t+\tau,j}).$$

The correlation matrix at lag τ is

$$\rho_\tau = [\mathrm{diag}(\boldsymbol{\Gamma}_0)]^{-1/2}\boldsymbol{\Gamma}_\tau[\mathrm{diag}(\boldsymbol{\Gamma}_0)]^{-1/2}, \qquad \boldsymbol{\Gamma}_{-\tau} = \boldsymbol{\Gamma}_\tau^\top.$$

White noise and VARMA process

Definition (Vector white noise)

$\{\varepsilon_t\}$ is white noise if

$$\mathbb{E}(\varepsilon_t) = \boldsymbol{\mu}, \qquad \mathrm{Cov}(\varepsilon_t, \varepsilon_{t+\tau}) = \begin{cases} \boldsymbol{\Gamma}_0, & \tau = 0, \\ \mathbf{0}, & \tau \neq 0. \end{cases}$$

We write $\varepsilon_t \sim \mathrm{WN}(\boldsymbol{\mu}, \boldsymbol{\Gamma}_0)$.

Definition (VARMA(p, q))

An N-dimensional process $\{\mathbf{x}_t\}$ is $\mathrm{VARMA}(p, q)$ if

$$\mathbf{x}_t = \boldsymbol{\alpha} + \Phi_1 \mathbf{x}_{t-1} + \cdots + \Phi_p \mathbf{x}_{t-p} + \varepsilon_t + \Theta_1 \varepsilon_{t-1} + \cdots + \Theta_q \varepsilon_{t-q},$$

where $\varepsilon_t \sim \mathrm{WN}(\mathbf{0}, \boldsymbol{\Gamma})$, $\Phi_i, \Theta_j \in \mathbb{R}^{N \times N}$, and $\Phi_p, \Theta_q \neq \mathbf{0}$.

Matrix polynomials:

$$\Phi(B) = \mathbf{I}_N - \Phi_1 B - \cdots - \Phi_p B^p, \qquad \Theta(B) = \mathbf{I}_N + \Theta_1 B + \cdots + \Theta_q B^q.$$

VARMA process (cont.): Theorem 4.4 and causality

Theorem (Theorem 4.4)

If $\mathbf{x}_t \sim \mathrm{VARMA}(p, q)$ *and all roots of*

$$\det(\mathbf{I}_N - \Phi_1 z - \cdots - \Phi_p z^p) = 0$$

lie outside the unit circle, then $\{\mathbf{x}_t\}$ *is weakly stationary.*

Definition (Causality)

$\{\mathbf{x}_t\}$ is causal if it admits a one-sided VMA(∞) representation

$$\mathbf{x}_t = \sum_{j=0}^{\infty} \Psi_j \varepsilon_{t-j}, \qquad \sum_{j=0}^{\infty} \|\Psi_j\| < \infty,$$

so $\mathbf{x}_t$ depends only on the current and past innovations.

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Yule-Walker estimator

For a mean-zero weakly stationary $\mathrm{VAR}(p)$,

$$\mathbf{x}_t = \Phi_1 \mathbf{x}_{t-1} + \cdots + \Phi_p \mathbf{x}_{t-p} + \varepsilon_t, \qquad \varepsilon_t \sim \mathrm{WN}(\mathbf{0}, \mathbf{\Gamma}),$$

the Yule-Walker equations are, for $\tau \geq 1$,

$$\mathbf{\Gamma}_\tau = \mathbf{\Gamma}_{\tau-1} \Phi_1^\top + \cdots + \mathbf{\Gamma}_{\tau-p} \Phi_p^\top,$$

and at lag 0,

$$\mathbf{\Gamma}_0 = \mathbf{\Gamma}_1^\top \Phi_1^\top + \cdots + \mathbf{\Gamma}_p^\top \Phi_p^\top + \mathbf{\Gamma}.$$

Hence $\tau = 1, \ldots, p$ gives p matrix equations for $\Phi_1, \ldots, \Phi_p$.

Yule-Walker estimator (cont.)

Define

$$\boldsymbol{\Gamma}(p) = \begin{pmatrix} \boldsymbol{\Gamma}_1^\top \\ \vdots \\ \boldsymbol{\Gamma}_p^\top \end{pmatrix}, \qquad \Phi(p) = \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_p \end{pmatrix},$$

and the block Toeplitz matrix

$$\boldsymbol{\Sigma}_p = \begin{pmatrix} \boldsymbol{\Gamma}_0 & \boldsymbol{\Gamma}_1^\top & \cdots & \boldsymbol{\Gamma}_{p-1}^\top \\ \boldsymbol{\Gamma}_1 & \boldsymbol{\Gamma}_0 & \cdots & \boldsymbol{\Gamma}_{p-2}^\top \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\Gamma}_{p-1} & \boldsymbol{\Gamma}_{p-2} & \cdots & \boldsymbol{\Gamma}_0 \end{pmatrix}.$$

Then

$$\Phi(p) = \boldsymbol{\Sigma}_p^{-1}\boldsymbol{\Gamma}(p), \qquad \boldsymbol{\Gamma} = \boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}(p)^\top \Phi(p).$$

For $\tau \geq 0$, estimate by

$$\widehat{\boldsymbol{\Gamma}}_\tau = \frac{1}{T}\sum_{t=1}^{T-\tau}(\mathbf{x}_t - \bar{\mathbf{x}})(\mathbf{x}_{t+\tau} - \bar{\mathbf{x}})^\top,$$

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Sample cross-correlation matrix (CCM)

For $\tau \geq 0$, start from the sample cross-covariance matrix

$$\widehat{\mathbf{\Gamma}}_\tau = \frac{1}{T} \sum_{t=1}^{T-\tau} (\mathbf{x}_t - \bar{\mathbf{x}})(\mathbf{x}_{t+\tau} - \bar{\mathbf{x}})^\top,$$

and use $\widehat{\mathbf{\Gamma}}_{-\tau} = \widehat{\mathbf{\Gamma}}_\tau^\top$ for negative lags. Define the sample cross-correlation matrix at lag τ by

$$\widehat{\rho}_\tau = [\text{diag}(\widehat{\mathbf{\Gamma}}_0)]^{-1/2} \widehat{\mathbf{\Gamma}}_\tau [\text{diag}(\widehat{\mathbf{\Gamma}}_0)]^{-1/2}.$$

Its (i, j) entry is

$$\widehat{\rho}_{ij,\tau} = \frac{\widehat{s}_{ij,\tau}}{\sqrt{\widehat{s}_{ii,0}\widehat{s}_{jj,0}}},$$

so diagonal entries are sample ACFs and off-diagonal entries are sample cross-correlations.

Sample cross-correlation matrix (cont.)

If $\{\mathbf{x}_t\} \sim \mathrm{VMA}(q)$ and $|\tau| > q$, then approximately

$$\mathrm{Var}(\widehat{\rho}_{ij,\tau}) \approx \frac{1}{T}\left(1 + 2\sum_{v=1}^{q} \rho_{v,ii}\rho_{v,jj}\right).$$

In the white-noise case ($q = 0$), a rough 95% reference band is

$$\pm\frac{1.96}{\sqrt{T}}.$$